ASYMPTOTIC EXPANSION OF THE OFF-DIAGONAL BERGMAN KERNEL ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT. We compute the first four coefficients of the asymptotic off-diagonal expansion in [17] of the Bergman kernel for the N-th power of a positive line bundle on a compact Kähler manifold, and we show that the coefficient b_1 of the $N^{-1/2}$ term vanishes when we use a K-frame. We also show that all the coefficients of the expansion are polynomials in the K-coordinates and the covariant derivatives of the curvature and are homogeneous with respect to the weight w in [12].

INTRODUCTION

The subject of this paper is the asymptotic expansion of the Bergman kernels for holomorphic sections of powers L^N of a positive holomorphic line bundle L on a compact Kähler manifold M. The Bergman kernel $K_N(z,w)$ for $L^N \to M$ is the kernel of the orthogonal projection from the space of $\mathcal{L}^2$ sections of L^N to the holomorphic sections. In 1990, Tian [25] gave the leading asymptotics of the diagonal kernel $K_N(z,z)$, and a complete asymptotic expansion was given by Zelditch [27] and independently by Catlin [6] using the Boutet de Monvel-Sjöstrand parametrix [5]. A purely complex-geometric proof of the Catlin-Zelditch expansion is given in [11]; see also [2,7,13] for other approaches. The first three terms of the diagonal expansion were computed in [12]; alternative approaches to the computation of the terms of the expansion are given in [8,26].

Zelditch [27] obtained the asymptotic expansion of $K_N(z,z)$ by viewing it as a Szegő kernel as follows: the holomorphic sections of L^N can be regarded as CR-holomorphic functions on a circle bundle $X \to M$ (as described below), and the kernels $K_N(z,w)$ lift to form the reproducing kernels $\Pi_N(x,y)$ for the Fourier components $\mathcal{H}_N^2(X)$ of the Hardy space $\mathcal{H}^2(X)$ of CR-holomorphic functions on X. Thus the $\Pi_N(x,y)$ are the Fourier components of the Szegő kernel for X.

The Bergman projection kernel $\Pi_N(z,w)$ can also be viewed as the covariance function (or two-point function) of a Gaussian random field on X. It was shown in [18] that the zero correlation current (which in one complex dimension gives the pair correlation for the point process of zeros of Gaussian holomorphic functions or sections) can be given by a universal formula depending only on the covariance $\Pi_N(z,w)$. This formula for the zero correlation current together with the scaled off-diagonal asymptotics of [17] was applied to the distribution of zeros, critical points and excursion sets of random holomorphic sections in [1,10,18–20,24]. For example, [18,19] generalize results of Sodin-Tsirelson [21] on the variance and asymptotic normality of the number of zeros of a random holomorphic function on a domain in $\mathbb{C}$; [20] generalizes [22] on the "hole probablity" of finding no zeros in a domain.

The off-diagonal asymptotics of the Bergman kernel have also found applications, for example, to random Bergman metrics [9], the convergence of geodesics in the space of Bergman metrics on toric varieties [23], and the asymptotics of spectral projectors associated to Toeplitz operators [16].

A complete asymptotic expansion of the scaled off-diagonal kernel on a compact, almost complex symplectic manifold M was given in [17]. In terms of normal coordinates in a point $z_0 \in M$, this expansion takes the form

$$N^{-m} \, \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) \sim \frac{1}{\pi^m} e^{u \cdot v - \frac{1}{2}(|u|^2 + |v|^2)}(1 + N^{-1/2}b_1 + N^{-1}b_2 + \cdots N^{-r/2}b_r + \cdots), \quad (1)$$

for $|u| + |v| < a$. (A precise statement is given in Theorem 1.2 below.)

In this paper, we consider the integrable case where M is a compact Kähler manifold. We show that the coefficients b_j of (1) are polynomials in the "K-coordinates" and the covariant derivatives of the curvature[1] (Theorem 1.7), and we compute the first few terms (Theorems 1.3 and 2.1). Our formula for computing the terms of the expansion is given in Lemma 2.6. We also show (Lemma 2.5) that the first coefficient b_1 of the expansion (1) vanishes when we use a "K-frame" for L and normal coordinates at z_0, and thus $N^{-m} \, \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})$ converges at the rate $1/N$ (instead of $1/\sqrt{N}$) as $N \to \infty$. The vanishing of b_1 (using different non-holomorphic coordinates and frame) was also given by Ma and Marinescu [13, (4.1.26)], [14, (2.19)].

The expansion (1) immediately gives the $\mathcal{C}^k$-bounds

$$D^k \log \Pi_N(u, v) = O(N^{\frac{k-1}{2}}), \quad \text{for} \quad |u| + |v| < \frac{a}{\sqrt{N}}, \quad k \geq 3. \quad (2)$$

(For $k = 1, 2$, the bound is $O(N^{\frac{k}{2}})$.) As a consequence of the proof of our formulas for the coefficients of (1), we prove that one can replace the bounds in (2) with the sharp bounds $O(\sqrt{N})$ for $k = 3$ and $O(N)$ for all $k \geq 4$ (Theorem 2.7).

1. STATEMENT OF RESULTS

Let M be a compact complex manifold polarized by an ample line bundle L. To describe the scaled off-diagonal asymptotic expansion, we give L a hermitian metric h with positive curvature Θ_h, and we give M the Kähler form $\omega = \frac{i}{2}\Theta_h$. We recall that the Hermitian metric h on L induces Hermitian metrics h^N on L^N, and we have

$$c_1(L^N, h^N) = N \, c_1(L, h) = \frac{N}{\pi} \, \omega.$$

The metrics h, ω induce Hermitian inner products on the (finite-dimensional) spaces $H^0(M, L^N)$ of holomorphic sections of L^N.

Let X be the unit circle bundle of the dual line bundle L^*. As in [3, 17, 27], we lift holomorphic sections $s \in H^0(M, L^N)$ to CR functions $\hat{s}$ on X satisfying $\hat{s}(e^{i\theta}x) = e^{iN\theta}\hat{s}(x)$. We denote the space of such functions by $\mathcal{H}^2_N(X)$. The Bergman kernel for $H^0(M, L^N)$ lifts to the orthogonal projector $\Pi_N : \mathcal{L}^2(X) \to \mathcal{H}^2_N(X)$, which is given by the *Szegő*

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[1]The curvature tensor itself is regarded as the 0-th covariant derivative. We use this convention throughout the paper.

kernel

$$\Pi_N(x,y) = \sum_{j=1}^{d_N} \widehat{S}_j^N(x)\overline{\widehat{S}_j^N(y)} \qquad (x,y \in X) \,,$$

where the $\widehat{S}_j^N$ form an orthonormal basis for $\mathcal{H}_N^2(X)$, and $d_N = \dim H^0(M, L^N)$.

We are interested in the scaled off-diagonal asymptotics of the Szegő kernel in a neighborhood of a point $z_0 \in M$. We let $z = (z_1, \ldots, z_m)$ denote local complex coordinates in a neighborhood U of z_0, and we let $|z| = \sqrt{|z_1|^2 + \cdots + |z_m|^2}$. Throughout this paper, we shall assume that the coordinates of z_0 are $(0, \ldots, 0) \in \mathbb{C}^m$. To properly describe the scaling asymptotics for the Szegő kernel at a point $z_0 \in M$, we need to use suitable coordinates in the circle bundle X. To do this, we first recall the following definition from [17]:

Definition 1.1. *A preferred local frame at a point $z_0 \in M$ is a $\mathcal{C}^\infty$ section $\mathbf{e}_L$ of L over a neighborhood U of z_0 such that $\bar{\partial}\mathbf{e}_L$ vanishes to first order at z_0 and*

$$\varphi(z) := -\log \|\mathbf{e}_L(z)\|_h^2 = \sum g_{j\bar{k}}(0)z_j\bar{z}_k + O(|z|^3) \,. \tag{3}$$

Suppose that $(z_1, \ldots, z_m)$ are complex coordinates on a neighborhood U of z_0 and $\mathbf{e}_L$ is a $\mathcal{C}^\infty$ local frame over U. We then give points $e^{i\theta-\varphi(z)/2}\,\mathbf{e}_L^*(z)$ of the circle bundle X (over U) the coordinates $(z_1, \ldots, z_m, \theta) \in \mathbb{C}^m \times (\mathbb{R}/2\pi\mathbb{Z})$, and we write

$$\Pi_N(z, \theta_1; w, \theta_2) = \Pi_N^{z_0}(z, \theta_1; w, \theta_2) = \Pi_N\left(e^{i\theta_1-\varphi(z)/2}\,\mathbf{e}_L^*(z)\,,\ e^{i\theta_2-\varphi(w)/2}\,\mathbf{e}_L^*(w)\right) \,. \tag{4}$$

We also let

$$\Pi_{BF}(z, \theta_1; w, \theta_2) = \frac{1}{\pi^m}e^{i(\theta_1-\theta_2)+z\cdot\bar{w}-\frac{1}{2}(|z|^2+|w|^2)} \tag{5}$$

denote the Szegő kernel for the Bargmann-Fock space of functions on $\mathbb{C}^m$ (see [3]).

We recall the off-diagonal asymptotics from [17] (see also [18, §5]):

Theorem 1.2. (cf. [17, Theorem 3.1]) *Let $(L, h) \to (M, \omega)$ be a positive line bundle over an m-dimensional compact Kähler manifold with Kähler form $\omega = \frac{i}{2}\Theta_h$. Let $\mathbf{e}_L$ be a preferred local frame for L and let $z_1, \ldots, z_m$ be complex coordinates about a point $z_0 \in M$ such that the Kähler potential $\varphi := -\log \|\mathbf{e}_L(z)\|_h^2 = |z|^2 + O(|z|^3)$. Then for all $k \in \mathbb{Z}^+$,*

$$N^{-m}\Pi_N^{z_0}\left(\frac{u}{\sqrt{N}}, \frac{\theta_1}{N}; \frac{v}{\sqrt{N}}, \frac{\theta_2}{N}\right)$$
$$= \Pi_{BF}(u, \theta_1; v, \theta_2)\left[1 + \sum_{r=1}^{k} N^{-r/2}b_r(u,v) + N^{-(k+1)/2}E_{Nk}(u,v)\right] \,, \tag{6}$$

where

- *each $b_r(u,v)$ is a polynomial (in $u, v, \bar{u}, \bar{v}$) of degree at most $5r$, and is an even polynomial if r is even, and odd if r is odd;*
- *for all $a \in \mathbb{R}^+$ and $j \geq 0$, there exists a positive constant C_{jka} such that*

$$|D^j E_{Nk}(u,v)| \leq C_{jka} \quad for \ \ |u| + |v| < a \,. \tag{7}$$

Here $|D^j F(u,v)|$ denotes the sum of the norms of the partial derivatives of F of order j at (u, v).

If one chooses a smoothly varying family of normal coordinates and preferred frames about points z in a neighborhood of z_0, then the polynomials b_k and remainders E_{Nk}

in (6) vary smoothly in z, and the (z, u, v)-derivatives are bounded independently of N; setting $u = v = 0$ in (6), one obtains the Catlin-Zelditch asymptotic expansion [6, 27],

$$N^{-m} \Pi_N(z, z) \sim \frac{1}{\pi^m} \left[1 + a_1(z) N^{-1} + a_2(z) N^{-2} + \cdots + a_r(z) N^{-r} + \cdots \right]. \qquad (8)$$

Precisely, for all $j, k \geq 0$, there exist constants C_{jk} such that

$$\left\| N^{-m} \Pi_N(z, z) - \pi^{-m} [1 + a_1(z) N^{-1} + \cdots + a_k(z) N^{-k}] \right\|_{\mathcal{C}^j(M)} \leq N^{-k-1} C_{jk}.$$

Note that we do not assume in Theorem 1.2 that the coordinates z_j are holomorphic. (Indeed, Theorem 1.2 is stated in [17] for almost complex, symplectic manifolds.) Note also that the point in M with coordinates $(u_1/\sqrt{N}, \ldots, u_m/\sqrt{N})$ depends on the $\mathcal{C}^\infty$ germ at z_0 of the coordinate map, and thus the polynomials $b_r(u, v)$ are not well-defined functions on $T_{z_0}(M)$, but depend on the higher jets of the coordinate map and the local frame. However, in our main results (Theorems 1.7 and 2.1), we use K-coordinates and a K-frame, which we define below. By the uniqueness properties of these coordinates (see the discussion following Definition 1.6), our coefficients are intrinsic.

Let $z_1, \cdots, z_m$ be holomorphic local coordinates about z_0 and let $\mathbf{e}_L$ be a holomorphic frame about z_0. Then the Kähler form is given by

$$\omega = \frac{i}{2} \partial \bar{\partial} \varphi = \frac{i}{2} g_{j\bar{k}} \, dz_j \wedge d\bar{z}_k. \qquad (9)$$

The components of the curvature tensor are given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}. \qquad (10)$$

(Throughout this paper, formulas are summed over repeated indices.) By our convention, the scalar curvature is $\rho = R_i{}^i{}_j{}^j = g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$.

The above formulas hold at z_0 under the following weaker conditions. If $\mathbf{e}_L$ is a preferred frame at z_0, then (9) holds at z_0; if $\mathbf{e}_L$ is holomorphic to infinite order at z_0, i.e. $\bar{\partial} \mathbf{e}_L$ vanishes to infinite order at z_0, then (9) holds to infinite order at z_0.

If the complex coordinates $z_1, \ldots, z_m$ are *holomorphic to infinite order at z_0*, i.e. the 1-forms $\bar{\partial} z_j$ vanish to infinite order at z_0, then (9) and (10) hold to infinite order at z_0.

A consequence of our formula for the coefficients of the expansion (Theorem 2.1) is that b_1 vanishes and b_2 is an expression in the curvature tensor when we use appropriate coordinates and frame:

Theorem 1.3. *Let $(L, h) \to (M, \omega)$ with $\omega = \frac{i}{2} \partial \bar{\partial} \varphi$ be as in Theorem 1.2. Let $\mathbf{e}_L$ be a holomorphic frame and let $z_1, \ldots, z_m$ be holomorphic coordinates such that the potential φ is of the form*

$$\varphi(z) = |z|^2 + \frac{1}{4} \left. \frac{\partial^4 \varphi}{\partial z_j \partial z_p \partial \bar{z}_k \partial \bar{z}_l} \right|_{z=0} z_j \bar{z}_k z_p \bar{z}_q + O(|z|^5). \qquad (11)$$

Then

$$N^{-m} \, \Pi_N^{z_0} \left(\frac{u}{\sqrt{N}}, \frac{\theta_1}{N}; \frac{v}{\sqrt{N}}, \frac{\theta_2}{N} \right)$$

$$= \Pi_{BF}(u, \theta_1; v, \theta_2) \Big[1 + N^{-1} \Big(\tfrac{1}{2}\rho + \tfrac{1}{8}R(u, \bar{u}, u, \bar{u}) + \tfrac{1}{8}R(v, \bar{v}, v, \bar{v}) - \tfrac{1}{4}R(u, \bar{v}, u, \bar{v}) \Big)$$

$$+ \, N^{-3/2} E_{N3}(u, v) \Big] \, ,$$

where E_{N3} satisfies the estimate (7) *for $k = 3$.*

Here, R denotes the curvature tensor $R(s, \bar{t}, u, \bar{v}) = R_{i\bar{j}k\bar{l}}(z_0)s_i\bar{t}_j u_k \bar{v}_l$ and $\rho = \rho(z_0)$ is the scalar curvature at z_0. Under the assumption of the theorem, the coordinates are normal coordinates at z_0, and therefore

$$R_{j\bar{k}p\bar{q}}(0) = -\frac{\partial^2 g_{j\bar{k}}}{\partial z_p \partial \bar{z}_q}(0) = -\frac{\partial^4 \varphi}{\partial z_j \partial z_p \partial \bar{z}_k \partial \bar{z}_l}(0) \, .$$

Condition (11) means that $z_1, \ldots, z_m$ and $\mathbf{e}_L$ are "K-coordinates with a K-frame of order 4" as in Definition 1.5. (See the discussion following Definition 1.6 for the existence of holomorphic K-coordinates and K-frames.) The terms of the off-diagonal expansion of the Szegő kernel depend on the choice of coordinates and are in general not tensors. After we posted the first version of this paper, Xiaonan Ma and George Marinescu applied their prior work [13, 14] to compute the N^{-1} term of an off-diagonal expansion using different (non-holomorphic) coordinates and frame (see [15]).

Setting $v = 0$, $\theta_2 = 0$ in Theorem 1.3, we have:

Corollary 1.4. *Under the hypotheses of Theorem 1.3, the off-diagonal Szegő kernel at the pair of points $z_0 + u/\sqrt{N}$ and z_0 is given by*

$$N^{-m} \, \Pi_N^{z_0} \left(\frac{u}{\sqrt{N}}, \frac{\theta}{N}; 0, 0 \right) = \frac{1}{\pi^m} e^{-\frac{1}{2}|u|^2 + i\theta} \Big[1 + N^{-1} \Big(\tfrac{1}{2}\rho + \tfrac{1}{8}R(u, \bar{u}, u, \bar{u}) \Big) + O(N^{-3/2}) \Big]$$

for $|u| < a$, where $a \in \mathbb{R}^+$.

Recall that $R(u, \bar{u}, u, \bar{u})$ is the holomorphic sectional curvature in the u-direction multiplied by $|u|^4$.

In order for the higher b_j to be well-defined, we need to strengthen the hypotheses on the coordinates and local frame. To do this we first summarize [11, Definition A.1, A.2] as follows:

Definition 1.5. *Let $\mathbf{e}_L$ be a holomorphic frame of L about z_0 and $z_1, \cdots, z_m$ be a holomorphic coordinate system about z_0. Let $\varphi(z)$ be the Kähler potential function given by* (3). *Let*

$$\varphi(z) \sim |z|^2 + \sum a_{JK} z^J \bar{z}^K \, , \quad |J| + |K| \geq 3$$

be the Taylor expansion of the Kähler potential, where we use the notation $J = (j_1, \ldots, j_m)$, $|J| = j_1 + \cdots + j_m$, $z^J = z_1^{j_1} \cdots z_m^{j_m}$. Let $3 \leq n \leq \infty$. The frame $\mathbf{e}_L$ is called a K-frame of order n if $a_{JK} = 0$ whenever $|J| + |K| \leq n$ and either $|J| = 0$ or $|K| = 0$. The coordinates $z_1, \cdots, z_m$ are called the K-coordinates of order n, if $a_{JK} = 0$ whenever $|J| + |K| \leq n$ and either $|J| = 1$ or $|K| = 1$.

We recall that the notation $f(z) \sim \sum c_{JK} z^J \bar{z}^K$ for an asymptotic series means that $f(z) = \sum_{|J|+|K| \leq n} c_{JK} z^J \bar{z}^K + O(|z|^{n+1})$, for all $n \in \mathbb{Z}^+$.

Combining the two parts of Definition 1.5, we have K-coordinates with a K-frame of order n precisely when

$$\varphi(z) = |z|^2 + \sum_{\substack{|J| \geq 2,\, |K| \geq 2 \\ |J|+|K| \leq n}} a_{JK} z^J \bar{z}^K \;+\; O(|z|^{n+1})\,. \tag{12}$$

In particular, normal coordinates are K-coordinates of order 3; a preferred frame is a K-frame of order 2.

If ω is only smooth, one cannot always find K-coordinates or a K-frame of order $n = \infty$ in the above definition, so we extend our definition of K-coordinates and a K-frame for this case:

Definition 1.6. *Let $\mathbf{e}_L$ be a smooth frame about z_0 such that $\bar{\partial}\mathbf{e}_L$ vanishes to infinite order at a point z_0, and let $z_1, \ldots, z_m$ be complex coordinates about z_0 such that $\bar{\partial}z_j$ vanishes to infinite order at $z_0 = 0$. We say that these are K-coordinates with a K-frame at z_0 if the Taylor expansion of the Kähler potential has the form*

$$\varphi(z) \sim |z|^2 + \sum a_{JK} z^J \bar{z}^K\,, \quad |J| \geq 2\,, \; |K| \geq 2\,, \tag{13}$$

Bochner [4] showed that if φ is real-analytic, then (after adding a pluriharmonic function to φ, i.e. by making a holomorphic change of the local frame $\mathbf{e}_L$) there exist unique holomorphic K-coordinates of infinite order (up to a unitary transformation) and a unique holomorphic K-frame of infinite order (up to a unit complex number), and the expansion (13) is in fact a convergent power series equal to $\varphi(z)$. When the Kähler form ω is $\mathcal{C}^\infty$, one can find holomorphic K-coordinates and a holomorphic K-frame of any finite order n. Furthermore, in this case, one can use Bochner's method to find formal power series solutions to (13). Using the fact that any formal power series is the asymptotic Taylor series of a $\mathcal{C}^\infty$ function, one obtains the existence of smooth K-coordinates with a K-frame (of infinite order) as in Definition 1.6.

The second main result of this paper, Theorem 1.7 below, is that the coefficients b_j of the asymptotic expansion (6) are "homogeneous" expressions in the curvature tensor. To describe what this means, we let $\mathcal{F}$ be the ring of polynomials of the covariant derivatives of the curvature with coefficients in $\mathbb{Q}$. A monomial in $\mathcal{F}$ is defined to be a product of terms like $\rho_{,I\bar{J}}$, $\mathrm{Ric}_{i\bar{j},I\bar{J}}$, or $R_{i\bar{j}k\bar{l},I\bar{J}}$, where $I = (i_1 \cdots i_p)$ and $J = (j_1 \cdots j_q)$ are multiple indices. In [12], the weight w is defined by:

$$w(\rho_{,I\bar{J}}) = w(\mathrm{Ric}_{i\bar{j},I\bar{J}}) = w(R_{i\bar{j}k\bar{l},I\bar{J}}) = 1 + \frac{p+q}{2}. \tag{14}$$

We let $\mathcal{P}$ be the space of polynomials of $u, v, \bar{u}, \bar{v}$ with coefficients in $\mathcal{F}$, and we extend the definition of w to monomials in $\mathcal{P}$ by requiring that $w(AB) = w(A) + w(B)$ and $w(u) = w(v) = w(\bar{u}) = w(\bar{v}) = 0$. We call an element in $\mathcal{P}$ w-homogeneous if all of its monomials have the same weight. If an element is w-homogeneous, we define its degree to be the weight of its monomials.

Theorem 1.7. *Let $2 \leq r \leq n - 2$. Then under the hypotheses of Theorem 1.2 with K-coordinates $(z_1, \ldots, z_m)$ and K-frame $\mathbf{e}_L$ of order n, the coefficient b_r is a polynomial in $u, v, \bar{u}, \bar{v}$ and the covariant derivatives of the curvature, and is w-homogeneous of weight $r/2$. Moreover, as a polynomial in $u, v, \bar{u}, \bar{v}$ (with coefficients in the ring $\mathcal{F}$ defined above):*

- *if r is even, b_r is an even polynomial of degree $2r$ in $u, v, \bar{u}, \bar{v}$;*
- *if r is odd, b_r is an odd polynomial of degree $2r - 1$ in $u, v, \bar{u}, \bar{v}$.*

Remark: Theorem 3.1 in [17] says that b_r is a polynomial of the same parity as r, but gives the bound $\deg b_r \leq 5r$ for the more general case of almost complex symplectic manifolds (see Theorem 1.2 above). We note that Theorem 1.7 states that b_r has exactly the degree $2r$ ($2r - 1$, respectively) for r even (odd, respectively), when the terms $\rho_{,I\bar{J}}$, $\mathrm{Ric}_{i\bar{j},I\bar{J}}$, $R_{i\bar{j}k\bar{l},I\bar{J}}$, etc., are regarded as abstract coefficients. When they are evaluated at a point $z_0 \in M$, then $2r, 2r - 1$ are only upper bounds for the degree of b_r. For example, if M is a polarized abelian variety with the flat Kähler metric, then all the b_r vanish.

2. COMPUTATION OF THE COEFFICIENTS

In this section, we prove Theorem 1.3 and also give formulas for the coefficients b_3 and b_4. We begin by describing notation that we use for these formulas. Let $z_1, \cdots, z_m$ be K-coordinates at $z_0 \in M$. Writing

$$\mathbf{u} = \sum u_j \, \partial/\partial z_j|_{z_0} \in T_{z_0}^{1,0}, \quad \overline{\mathbf{u}} = \sum \bar{u}_j \, \partial/\partial \bar{z}_j|_{z_0} \in T_{z_0}^{0,1},$$

$$\mathbf{v} = \sum v_j \, \partial/\partial z_j|_{z_0} \in T_{z_0}^{1,0}, \quad \overline{\mathbf{v}} = \sum \bar{v}_j \, \partial/\partial \bar{z}_j|_{z_0} \in T_{z_0}^{0,1},$$

we let

$$
\begin{aligned}
S(\mathbf{u}, \overline{\mathbf{v}}) &= -R(\mathbf{u}, \overline{\mathbf{v}}, \mathbf{u}, \overline{\mathbf{v}}) = -R_{i\bar{j}k\bar{l}} u_i u_k \bar{v}_j \bar{v}_l \,, \\
L(\mathbf{u}, \overline{\mathbf{v}}) &= -R_{i\bar{j}k\bar{l},s} u_i u_k u_s \bar{v}_j \bar{v}_l - R_{i\bar{j}k\bar{l},\bar{s}} u_i u_k \bar{v}_s \bar{v}_j \bar{v}_l, \\
K_1(\mathbf{u}, \overline{\mathbf{v}}) &= (-R_{i\bar{j}k\bar{l},s\bar{t}} + R_{p\bar{j}k\bar{l}} R_{i\bar{p}s\bar{t}} + R_{i\bar{j}p\bar{l}} R_{k\bar{p}s\bar{t}} + R_{i\bar{p}k\bar{t}} R_{p\bar{j}s\bar{l}}) u_i u_k u_s \bar{v}_j \bar{v}_l \bar{v}_t \,, \\
K_2(\mathbf{u}, \overline{\mathbf{v}}) &= -R_{i\bar{j}k\bar{l},st} u_i u_k u_s u_t \bar{v}_j \bar{v}_l - R_{i\bar{j}k\bar{l},\bar{s}\bar{t}} u_i u_k \bar{v}_j \bar{v}_l \bar{v}_s \bar{v}_t.
\end{aligned}
\tag{15}
$$

Recall that the covariant derivatives of tensors, such as those appearing in (15), are defined as

$$T_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q, s} = \frac{\partial T_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}}{\partial z_s} - \sum_{t=1}^{p} \Gamma_{i_t s}^{r} T_{i_1 \cdots r \cdots i_p \bar{j}_1 \cdots \bar{j}_q},$$

$$T_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q, \bar{s}} = \frac{\partial T_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}}{\partial \bar{z}_s} - \sum_{t=1}^{q} \overline{\Gamma_{j_t s}^{r}} T_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{r} \cdots \bar{j}_q}.$$

For any function $f : T_{z_0}^{1,0} \times T_{z_0}^{0,1} \to \mathbb{C}$, we let $f^{\sharp} : T_{z_0}^{1,0} \times T_{z_0}^{1,0} \to \mathbb{C}$ be the function given by

$$f^{\sharp}(\mathbf{u}, \mathbf{v}) = f(\mathbf{u}, \overline{\mathbf{v}}) - \tfrac{1}{2} f(\mathbf{u}, \overline{\mathbf{u}}) - \tfrac{1}{2} f(\mathbf{v}, \overline{\mathbf{v}}), \qquad \mathbf{u}, \mathbf{v} \in T_{z_0}^{1,0}. \tag{16}$$

We now can state our formulas for the first 4 coefficients of the asymptotic expansion of Theorem 1.2:

Theorem 2.1. *Let $(L, h) \to (M, \omega)$ be a positive line bundle over a compact Kähler manifold with Kähler form $\omega = \frac{i}{2}\Theta_h$. Let $z_1, \ldots, z_m$ be K-coordinates with a K-frame $\mathbf{e}_L$ at a point $z_0 \in M$ (as in Definition 1.6).*

Then the first four coefficients in the off-diagonal asymptotic expansion (6) *are given by*

$$b_1(u, v) = 0,$$

$$b_2(u, v) = \frac{1}{2}\rho + \frac{1}{4}S^\sharp(\mathbf{u}, \mathbf{v}),$$

$$b_3(u, v) = \frac{1}{2}\nabla\rho(\mathbf{u} + \overline{\mathbf{v}}) + \frac{1}{12}L^\sharp(\mathbf{u}, \mathbf{v}),$$

$$b_4(u, v) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2) + \frac{1}{4}\nabla^2\rho(\mathbf{u} + \overline{\mathbf{v}}, \mathbf{u} + \overline{\mathbf{v}})$$

$$+ \frac{1}{36}K_1^\sharp(\mathbf{u}, \mathbf{v}) + \frac{1}{48}K_2^\sharp(\mathbf{u}, \mathbf{v}) + \frac{1}{8}\left(\rho + \frac{1}{2}S^\sharp(\mathbf{u}, \mathbf{v})\right)^2,$$

where ρ, Ric, R *are the scalar curvature, Ricci curvature tensor, and the curvature tensor, respectively, at* z_0, *and* $S^\sharp, L^\sharp, K_1^\sharp, K_2^\sharp$ *are given by* (15)–(16). *Furthermore, the formula for* b_j *holds when* $z_1, \ldots, z_m$ *are K-coordinates with a K-frame* $\mathbf{e}_L$ *of order* $j + 2$ *(as in Definition 1.5), for* $1 \leq j \leq 4$.

The values of b_1, b_2 in Theorem 2.1 yield the formula of Theorem 1.3. Theorem 2.1 also provides scaling asymptotics of the *normalized Szegő kernel* in [18]:

$$P_N(z, w) := \frac{|\Pi_N(z, 0; w, 0)|}{\Pi_N(z, 0; z, 0)^{\frac{1}{2}} \Pi_N(w, 0; w, 0)^{\frac{1}{2}}}. \tag{17}$$

Corollary 2.2. *Under the hypotheses and notation of Theorem 2.1,*

$$P_N\left(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}\right) = e^{-\frac{1}{2}|u-v|^2}\left[1 + \frac{1}{4}\mathrm{Re}\, S^\sharp(\mathbf{u}, \mathbf{v})\, N^{-1} + \frac{1}{12}\mathrm{Re}\, L^\sharp(\mathbf{u}, \mathbf{v})\, N^{-3/2} + O(N^{-2})\right].$$

We begin the proof of Theorem 2.1 by introducing some more notation:

Definition 2.3. *For a local smooth frame* $\mathbf{e}_L$ *of* $L \to M$ *over a trivializing neighborhood* U, *we define the kernel* $B_N(z, w)$ *on* $U \times U$ *by*

$$\Pi_N(z, \theta_1; w, \theta_2) = e^{iN(\theta_1 - \theta_2)}e^{-\frac{N}{2}\varphi(z) - \frac{N}{2}\varphi(w)}B_N(z, w).$$

Note that if we write $S_j^N = f_j\mathbf{e}_L^N$, where $\{S_j^N\}$ is an orthonormal basis for $H^0(M, L^N)$, then

$$B_N(z, w) = \sum f_j^N(z)\overline{f_j^N(w)}. \tag{18}$$

In particular, $B_N(z, w)$ is independent to θ_1, θ_2.

If $\mathbf{e}_L$ is holomorphic, then the f_j^N are holomorphic and thus $B_N(z, w)$ is holomorphic in z and anti-holomorphic in w. More generally, if $\bar\partial\mathbf{e}_L$ vanishes to infinite order at z_0, then $B_N(z, w)$ is holomorphic in z and anti-holomorphic in w to infinite order at (z_0, z_0).

To simplify notation, we now write

$$\Pi_N(z, w) = \Pi_N^{z_0}(z, 0; w, 0), \quad i.e. \quad \Pi_N^{z_0}(z, \theta_1; w, \theta_2) = e^{iN(\theta_1 - \theta_2)}\Pi_N(z, w),$$

$$\varphi_{P\overline{Q}}(z) = \frac{\partial^{|P|+|Q|}\varphi(z)}{\partial z^P \partial\bar z^Q}.$$

For a multi-index $P = (p_1, \cdots, p_m)$, we define $P! = p_1! \cdots p_m!$.

Lemma 2.4. *Suppose that $(z_1, \ldots, z_m)$ are normal coordinates holomorphic to infinite order at z_0, and $\mathbf{e}_L$ is a preferred frame holomorphic to infinite order at z_0.*

Then for all $k \in \mathbb{Z}^+$ and $a \in \mathbb{R}^+$, there exists a positive constant C_{ka} such that

$$\log \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = \sum_{|P|+|Q| \leq k-1} \frac{1}{P!Q!} N^{-\frac{|P|+|Q|}{2}} \left.\frac{\partial^{|P|+|Q|} \log \Pi_N(z,z)}{\partial z^P \partial \bar{z}^Q}\right|_{z=0} u^P \bar{v}^Q$$

$$+ \sum_{|P|+|Q| \leq k+1} \frac{1}{P!Q!} N^{-\frac{|P|+|Q|}{2}+1} \varphi_{P\overline{Q}}(0) \left[u^P \bar{v}^Q - \frac{1}{2} u^P \bar{u}^Q - \frac{1}{2} v^P \bar{v}^Q \right]$$

$$+ N^{-\frac{k}{2}} F_{Nk}(u,v), \tag{19}$$

where $|F_{Nk}(u,v)| \leq C_{ka}$ for $|u| + |v| < a$, for all $N > 0$.

Proof. By Definition 2.3 and the simplified notation after the definition,

$$\log \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = \log B_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) - \frac{N}{2} \varphi(\frac{u}{\sqrt{N}}) - \frac{N}{2} \varphi(\frac{v}{\sqrt{N}}). \tag{20}$$

By (18), $B_N(u,v)$ is holomorphic in u and anti-holomorphic in v to infinite order at $(0,0)$, and therefore by Taylor's formula,

$$\left| \log B_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) - \sum_{|P|+|Q| \leq 5k-1} \frac{1}{P!Q!} N^{-\frac{|P|+|Q|}{2}} \left.\frac{\partial^{|P|+|Q|} \log B_N(z,z)}{\partial z^P \partial \bar{z}^Q}\right|_{z=0} u^P \bar{v}^Q \right|$$

$$\leq C_{mk} \left(\frac{a}{\sqrt{N}}\right)^{5k} \sup_{|u|+|v| < \frac{a}{\sqrt{N}}} |D^{5k} \log B_N(u,v)|. \tag{21}$$

Next, we estimate $D^k \log B_N(u,v)$: by Theorem 1.2,

$$\log \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = u \cdot \bar{v} - \tfrac{1}{2}|u|^2 - \tfrac{1}{2}|v|^2 + \sum_{j=1}^{k-1} N^{-j/2} \beta_j(u,v) + N^{-k/2} \widetilde{F}_{Nk}(u,v), \tag{22}$$

where $\beta_1, \ldots, \beta_{k-1}$ are polynomials in $(u, \bar{u}, v, \bar{v})$, and $|D^j \widetilde{F}_{Nk}(u,v)| < \widetilde{C}_{jka}$ for $|u|+|v| < a$.

Claim: $\deg \beta_j \leq 5j$.

To verify the claim, we recall from Theorem 1.2 that $\deg b_r \leq 5r$. Assigning the weights $\nu(N^{-1/2}) = -5$, $\nu(u) = \nu(\bar{u}) = \nu(v) = \nu(\bar{v}) = 1$, and extending the definition of ν to monomials in $N^{-1/2}, u, \bar{u}, v, \bar{v}$ by requiring that $\nu(AB) = \nu(A) + \nu(B)$, we see that all the monomials in (6) have ν-weights ≤ 0. Since the terms $N^{-j/2} \beta_j$ in (22) are polynomials in the $N^{-r/2} b_r$, the monomials in $N^{-j/2} \beta_j$ also have ν-weights ≤ 0, which verifes the claim.

Since $u \cdot \bar{v} - \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2 + \sum_{j=1}^{k-1} N^{-j/2} \beta_j(u,v)$ is a polynomial of degree $\leq 5k - 5$, we change variables from $(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})$ to (u,v) in (22) to conclude that

$$|D^{5k} \log \Pi_N(u,v)| = N^{-k/2}|D^{5k} \widetilde{F}_{Nk}(\sqrt{N}\,u, \sqrt{N}\,v)| \leq \widetilde{C}_{[5k]ka} N^{2k}, \quad \text{for } |u| + |v| < \frac{a}{\sqrt{N}}. \tag{23}$$

By (20), $\log B_N(z, z) = \log \Pi_N(z, z) + N\varphi(z)$. Therefore by (21) and (23),

$$\log B_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})$$

$$= \sum_{|P|+|Q|\leq 5k-1} \frac{1}{P!Q!} N^{-\frac{|P|+|Q|}{2}} \frac{\partial^{|P|+|Q|} \log B_N(z,z)}{\partial z^P \partial \bar{z}^Q}\Big|_{z=0} u^P \bar{v}^Q + N^{-k/2} E^1_{Nka}(u, v)$$

$$= \sum_{|P|+|Q|\leq 5k-1} \frac{1}{P!Q!} N^{-\frac{|P|+|Q|}{2}} \left[\frac{\partial^{|P|+|Q|} \log \Pi_N(z,z)}{\partial z^P \partial \bar{z}^Q}\Big|_{z=0} + N\varphi_{P\overline{Q}}(0) \right] u^P \bar{v}^Q$$

$$+ N^{-k/2} E^1_{Nka}(u, v), \tag{24}$$

where $|E^1_{Nka}(u, v)| \leq C_{ka}$ for $|u| + |v| < a$. Furthermore, by (20) and Taylor's formula,

$$\log \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = \log B_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})$$

$$- \frac{N}{2} \sum_{|P|+|Q|\leq k+2} N^{-\frac{|P|+|Q|}{2}} \varphi_{P\overline{Q}}(0)(u^P \bar{u}^Q + v^P \bar{v}^Q) + N^{-\frac{k+1}{2}} E^2_{Nka}(u, v), \tag{25}$$

where $|E^2_{Nka}(u, v)| \leq C_{ka}$ for $|u| + |v| < a$.

Since the derivatives of $\log \Pi_N$ (of order > 0) converge uniformly to 0 on the diagonal (by the Catlin-Zelditch asymptotic expansion (8)), each of the terms $\frac{\partial^{|P|+|Q|} \log \Pi_N(z,z)}{\partial z^P \partial \bar{z}^Q}|_{z=0}$ is bounded independently of N. Hence, we can move the $O(N^{-k/2})$ terms in the summation in (24) to the remainder. Applying (25), we then obtain the formula of the lemma. $\square$

We now give a quick proof that the first term b_1 of the expansion (6) vanishes. (This fact also follows from formula (27) below for the coefficients β_1 of (22).)

Lemma 2.5. *If $(z_1, \ldots, z_m)$ are normal coordinates about z_0 and $\mathbf{e}_L$ is a K-frame of order 3, then*

$$b_1(u, v) = 0.$$

Proof. Under the assumptions on the coordinates and frame, we have

$$\varphi(z, \bar{w}) = z \cdot \bar{w} + O(|z| + |w|)^4.$$

Thus by Lemma 2.4 with $k = 2$,

$$\log \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = \log \Pi_N(0, 0) + \frac{1}{\sqrt{N}} \left(u_j \frac{\partial}{\partial z_j} + \bar{v}_j \frac{\partial}{\partial \bar{z}_j} \right) \log \Pi_N(z, z)\Big|_{z=0}$$

$$+ u \cdot \bar{v} - \tfrac{1}{2}|u|^2 - \tfrac{1}{2}|v|^2 + O(\frac{1}{N}).$$

On the other hand, by (5)–(6)

$$\log \Pi_N(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}}) = m \log \frac{N}{\pi} + u \cdot \bar{v} - \tfrac{1}{2}|u|^2 - \tfrac{1}{2}|v|^2 + N^{-1/2} b_1(u, v) + O(\frac{1}{N}).$$

But by the Catlin-Zelditch asymptotics (8), $d \log \Pi_N(z, z) = O(\frac{1}{N})$, and therefore $b_1 = 0$. $\square$

Taking the logarithm of (8), we obtain a $\mathcal{C}^\infty$ asymptotic expansion of the form

$$\log \Pi_N(z, z) \sim m \log(N/\pi) + \alpha_1(z)N^{-1} + \alpha_2(z)N^{-2} + \cdots + \alpha_r(z)N^{-r} + \cdots. \tag{26}$$

Substituting this expansion in (19) and equating coefficients with those in (22), we obtain:

Lemma 2.6. *The coefficients β_j of (22) ($j \geq 1$) are polynomials of degree at most $j + 2$ given by*

$$\beta_j(u,v) = \sum_{r=1}^{\lfloor j/2 \rfloor} \sum_{|P|+|Q|=j-2r} \frac{1}{P!Q!} \frac{\partial^{|P|+|Q|}\alpha_r}{\partial z^P \partial \bar{z}^Q}(0)\, u^P \bar{v}^Q$$

$$+ \sum_{|P|+|Q|=j+2} \frac{1}{P!Q!} \varphi_{P\overline{Q}}(0) \left[u^P \bar{v}^Q - \frac{1}{2} u^P \bar{u}^Q - \frac{1}{2} v^P \bar{v}^Q \right], \qquad (27)$$

where the α_r are given by (26).

Remark: The estimate (23) is not sharp. A sharp estimate is given by the following result of independent interest.

Theorem 2.7. *There exist positive constants C_{ka} for $k \in \mathbb{Z}^+, a \in \mathbb{R}^+$, such that under the hypotheses of Lemma 2.4,*

$$|D^k \log \Pi_N(u,v)| \leq C_{ka} N, \quad for\ |u| + |v| < a/\sqrt{N},\ k \in \mathbb{Z}^+.$$

Furthermore,

$$|D^1 \log \Pi_N(u,v)| + |D^3 \log \Pi_N(u,v)| \leq C_a \sqrt{N}, \quad for\ |u| + |v| < a/\sqrt{N}.$$

Proof. Let β_j be as in (22). We have by (22) and Lemma 2.5,

$$\log \Pi_N(u,v) = m \log \frac{N}{\pi} + N(u \cdot \bar{v} - \tfrac{1}{2}|u|^2 - \tfrac{1}{2}|v|^2) + N^{-1}\widetilde{E}_{N2}(\sqrt{N}\,u, \sqrt{N}\,v).$$

Differentiating the above equation up to 3 times, we obtain the result for $k \leq 3$. For $k \geq 4$, by Lemma 2.6,

$$\deg \left[u \cdot \bar{v} - \tfrac{1}{2}|u|^2 - \tfrac{1}{2}|v|^2 + \sum_{j=1}^{k-3} N^{-j/2}\beta_j(u,v) \right] \leq k - 1.$$

As in the proof of Lemma 2.4, we apply (22) (with k replaced by $k - 2$) to conclude that

$$|D^k \log \Pi_N(u,v)| = N^{-\frac{k-2}{2}} \left| D^k \left[\widetilde{E}_{N[k-2]}(\sqrt{N}\,u, \sqrt{N}\,v) \right] \right| \leq C_{ka} N, \quad for\ |u| + |v| < a/\sqrt{N}.$$

$\square$

Example 1: Let L be the hyperplane section bundle over $\mathbb{CP}^m$ with the Fubini-Study metric. Let $z_j = Z_j/Z_0$ for $1 \leq j \leq m$, where $(Z_0 : \cdots : Z_m)$ are the homogeneous coordinates in $\mathbb{CP}^m$; then $z_1, \ldots, z_m$ are K-coordinates at $z_0 = (1 : 0 : \cdots : 0)$. In terms of these K-coordinates and a K-frame at z_0, we have

$$\log \Pi_N(u,v) = \log \frac{(N+m)!}{\pi^m N!} + N \log(1 + u \cdot \bar{v}) - \frac{N}{2} \log(1 + |u|^2) - \frac{N}{2} \log(1 + |v|^2). \quad (28)$$

(See for example [3].) Differentiating, we see that

$$\sup_{|u|+|v|<a/\sqrt{N}} |D^{2j} \log \Pi_N(u,v)| \sim c_{2j}\, N, \qquad \sup_{|u|+|v|<a/\sqrt{N}} |D^{2j-1} \log \Pi_N(u,v)| \sim c_{2j-1}\, \sqrt{N},$$

where the c_k are constants depending on a. This example shows that Proposition 2.7 is sharp for k even and for $k = 1, 3$.

Replacing (u,v) with $(\frac{u}{\sqrt{N}}, \frac{v}{\sqrt{N}})$ in (28) and exponentiating, we obtain the expansion

$$\Pi_N(\tfrac{u}{\sqrt{N}}, \tfrac{v}{\sqrt{N}}) = \Pi_{BF}(u,0;v,0)\left[1 + N^{-1}\left(\tfrac{m(m+1)}{2} - \tfrac{(u\cdot\bar{v})^2}{2} + \tfrac{|u|^4+|v|^4}{4}\right) + \cdots\right]. \qquad (29)$$

The expansion (29) can be used to check Theorem 1.3 for the case where $M = \mathbb{CP}^m$. To do this, the reader can substitute in Theorem 1.3 the values

$$R_{\bar{j}j\bar{j}j} = 2\,, \quad R_{j\bar{j}k\bar{k}} = R_{j\bar{k}k\bar{j}} = 1 \ (j \neq k)\,, \quad \text{others} = 0\,,$$

of the curvature tensor at z_0, and verify that the resulting expansion coincides with (29).

Example 2: To provide an example where Theorem 2.7 is sharp for all $k \geq 4$, we give M a Kähler metric with potential

$$\varphi = |z|^2 + \sum_{j=2}^{\infty}(z_1^2 \bar{z}_1^j + z_1^j \bar{z}_1^2)$$

in a neighborhood of $z_0 = 0$. By Lemma 2.6,

$$\beta_k(u,0) = -\frac{1}{2}\left(u_1^2 \bar{u}_1^k + u_1^k \bar{u}_1^2\right) + \quad \text{terms of lower degree.}$$

Furthermore $\beta_2, \ldots, \beta_{k-1}$ are of degree $\leq k+1$. Therefore by (22),

$$\frac{\partial^{k+2}}{\partial u_1^2 \partial \bar{u}_1^k} \log \Pi_N(\tfrac{u}{\sqrt{N}}, 0) = -\frac{1}{2}N^{-k/2} + O(N^{-(k+1)/2})\,.$$

It follows that $|D^{k+2} \log \Pi_N(u,0)| \sim N$ for $k \geq 3$ (and also for $k = 2$).

In order to use Lemma 2.6 to compute the β_j's (and then the b_j's), we need to express the derivatives of the Kähler potential φ in terms of the covariant derivatives of the curvature. Since we are using K-coordinates with a K-frame, each monomial in the expansion (13) has at least two z_j's and two $\bar{z}_j$'s. Therefore, to compute $\beta_2, \beta_3, \beta_4$, we need the derivatives of φ in the following lemma.

Lemma 2.8. *Using K-coordinates about $z_0 \in M$, we have the following identities at the point z_0:*

$$\begin{aligned}
&\text{i)} \quad \varphi_{ik\bar{j}\bar{l}} = -R_{i\bar{j}k\bar{l}} \\
&\text{ii)} \quad \varphi_{iks\bar{j}\bar{l}} = -R_{i\bar{j}k\bar{l},s} \\
&\text{iii)} \quad \varphi_{ik\bar{j}\bar{l}\bar{t}} = -R_{i\bar{j}k\bar{l},\bar{t}} \\
&\text{iv)} \quad \varphi_{iks\bar{j}\bar{l}\bar{t}} = -R_{i\bar{j}k\bar{l},s\bar{t}} + R_{p\bar{j}k\bar{l}}R_{i\bar{p}s\bar{t}} + R_{i\bar{j}p\bar{l}}R_{k\bar{p}s\bar{t}} + R_{i\bar{p}k\bar{t}}R_{p\bar{j}s\bar{l}} \\
&\text{v)} \quad \varphi_{ikst\bar{j}\bar{l}} = -R_{i\bar{j}k\bar{l},st} \\
&\text{vi)} \quad \varphi_{ik\bar{j}\bar{l}\bar{s}\bar{t}} = -R_{i\bar{j}k\bar{l},\bar{s}\bar{t}}
\end{aligned}$$

Furthermore, (i)–(iii) hold for K-coordinates of order 3 (i.e., normal coordinates), and (iv)–(vi) hold for K-coordinates of order 4.

Proof. Equation (i) follows from (10). Differentiating (10), we have

$$\begin{aligned}
\varphi_{iks\bar{j}\bar{l}} &= -\frac{\partial R_{i\bar{j}k\bar{l}}}{\partial z_s} + \frac{\partial}{\partial z_s}\left(g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}\right) \\
&= -R_{i\bar{j}k\bar{l},s} - \Gamma_{is}^p R_{p\bar{j}k\bar{l}} - \Gamma_{ks}^p R_{i\bar{j}p\bar{l}} + \frac{\partial}{\partial z_s}\left(g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z_k}\frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}\right), \qquad (30)
\end{aligned}$$

where Γ^i_{jk} are the Christoffel symbols,

$$\Gamma^i_{jk} = g^{i\bar{l}}\frac{\partial g_{j\bar{l}}}{\partial z_k}.$$

Thus equation (ii) follows by evaluating (30) at z_0. Equation (iii) is the conjugate of (ii). We recall that

$$\frac{\partial g_{i\bar{q}}}{\partial z_k}(z_0) = \frac{\partial^2 g_{i\bar{q}}}{\partial z_s \partial z_k}(z_0) = 0$$

for K-coordinates of order 4. Equation (v) then follows by applying $\partial/\partial z_t$ to (30) and evaluating at z_0; equation (vi) is the conjugate of (v).

Finally, by applying $\partial/\partial\bar{z}_t$ to (30), we have

$$\varphi_{ik s\bar{j}\bar{l}t}(z_0) = \left[-R_{i\bar{j}k\bar{l},s\bar{t}} - \varphi_{is\bar{p}\bar{t}}\,R_{p\bar{j}k\bar{l}} - \varphi_{ks\bar{p}\bar{t}}\,R_{i\bar{j}p\bar{l}} + \varphi_{ik\bar{p}\bar{t}}\,\varphi_{ps\bar{j}\bar{l}}\right]_{z=z_0}$$

and using (10) again, we get (iv). $\qquad\square$

Proof of Theorem 2.1. By [12, Theorem 1.1],[2]

$$\Pi_N(z,z) = \frac{N^m}{\pi^m}\left[1 + \frac{1}{2}\rho(z)\,N^{-1} + \left(\frac{1}{3}\Delta\rho + \frac{1}{24}|R|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}\rho^2\right)\bigg|_z N^{-2} + O(N^{-3})\right].$$
(31)

Therefore,

$$\alpha_1 = \frac{1}{2}\rho, \quad \alpha_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}|R|^2 - \frac{1}{6}|\mathrm{Ric}|^2.$$
(32)

By Lemma 2.5, $\beta_1 = 0$. To compute $\beta_2, \beta_3, \beta_4$, we write

$$\xi_{P\overline{Q}} = u^P v^{\overline{Q}}, \quad \xi^\sharp_{P\overline{Q}} = u^P v^{\overline{Q}} - \frac{1}{2}u^P u^{\overline{Q}} - \frac{1}{2}v^P v^{\overline{Q}}.$$

Expanding Lemma 2.6, we have

$$\begin{aligned}
\beta_2 &= \alpha_1 + \frac{1}{4}\,\varphi_{ik\bar{j}\bar{l}}\,\xi^\sharp_{ik\bar{j}\bar{l}}, \\
\beta_3 &= \frac{\partial\alpha_1}{\partial z_j}\,u_j + \frac{\partial\alpha_1}{\partial\bar{z}_j}\,\bar{v}_j + \frac{1}{12}\varphi_{ikp\bar{j}\bar{l}}\,\xi^\sharp_{ikp\bar{j}\bar{l}} + \frac{1}{12}\varphi_{ik\bar{p}\bar{j}\bar{l}}\,\xi^\sharp_{ik\bar{p}\bar{j}\bar{l}}, \\
\beta_4 &= \alpha_2 + \frac{\partial^2\alpha_1}{\partial z_j \partial\bar{z}_k}\,u_j\bar{v}_k + \frac{1}{2}\frac{\partial^2\alpha_1}{\partial z_j \partial z_k}\,u_j v_k + \frac{1}{2}\frac{\partial^2\alpha_1}{\partial\bar{z}_j \partial\bar{z}_k}\,\bar{u}_j\bar{v}_k \\
&\quad + \frac{1}{48}\,\varphi_{ikpq\bar{j}\bar{l}}\,\xi^\sharp_{ikpq\bar{j}\bar{l}} + \frac{1}{48}\,\varphi_{ik\bar{p}\bar{q}\bar{j}\bar{l}}\,\xi^\sharp_{ik\bar{p}\bar{q}\bar{j}\bar{l}} + \frac{1}{36}\varphi_{ikp\bar{q}\bar{j}\bar{l}}\,\xi^\sharp_{ikp\bar{q}\bar{j}\bar{l}}.
\end{aligned}$$

The above formula for β_j holds for K-coordinates with K-frame of order $j+2$ $(j = 2, 3, 4)$. Applying Lemma 2.8, we then obtain

$$\beta_2 = \frac{1}{2}\rho + \frac{1}{4}S^\sharp(\mathbf{u},\mathbf{v}),$$

$$\beta_3 = \frac{1}{2}\nabla\rho(\mathbf{u}+\bar{\mathbf{v}}) + \frac{1}{12}L^\sharp(\mathbf{u},\mathbf{v}),$$

$$\beta_4 = \frac{1}{3}\Delta\rho + \frac{1}{24}|R|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{4}\nabla^2\rho(\mathbf{u}+\bar{\mathbf{v}}, \mathbf{u}+\bar{\mathbf{v}}) + \frac{1}{36}K^\sharp_1(\mathbf{u},\mathbf{v}) + \frac{1}{48}K^\sharp_2(\mathbf{u},\mathbf{v}).$$

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[2]The Kähler form we use is π^{-1} times the Kähler form in [12]. Therefore the expansion (31) differs from the one in [12] by a factor π^{-m}.

Substituting these values of β_j in (22) and exponentiating, we obtain the formulas of Theorem 2.1. □

Remark: The coefficients b_5, b_6 can also be computed from Lemma 2.6 using the formula for α_3 obtained from [12].

3. Homogeneity of the coefficients

We now prove Theorem 1.7. Recall that $\mathcal{F}$ is the ring of polynomials of the covariant derivatives of the curvature with coefficients in $\mathbb{Q}$, and a polynomial is w-homogeneous if all of its monomials have the same w-weight, as given by (14). For a w-homogeneous polynomial, its w-weight is defined as the w-weight of any of its monomials.

Lemma 3.1. *Using coordinates $z_1, \ldots, z_m$ that are holomorphic to infinite order at z_0, the expressions*

$$\frac{\partial^{P+Q}\varphi}{\partial z^P \partial \bar{z}^Q} \, , \tag{33}$$

evaluated at z_0, are polynomials in

$$g_{i\bar{j}}, g^{i\bar{j}}, R_{i\bar{j}k\bar{l},A\bar{B}}, \frac{\partial^{|C|}g_{i\bar{j}}}{\partial \bar{z}^C}, \frac{\partial^{|D|}g_{i\bar{j}}}{\partial z^D}, \tag{34}$$

where $|P|, |Q| \geq 2$, $1 \leq |C| \leq |Q| - 2, 2 \leq |D| \leq |P| - 2$. Moreover, if $z_1, \ldots, z_m$ are K-coordinates of order $|P| + |Q|$ at the point z_0, then the expression (33), evaluated at z_0, is a w-homogeneous polynomial in $\mathcal{F}$ of w-weight $(|P| + |Q|)/2 - 1$.

Proof. We first observe that we can replace $\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}$ by $g_{i\bar{j}}$. Let $\mathcal{R}$ be the ring generated by the elements in (34).

We need to prove that $\mathcal{R}$ is closed under partial derivatives and at the same time, the w-homogeneity is kept. We use mathematical induction on $n = |P| + |Q|$. If $n = 4$, then the conclusion is obvious because $|P| = |Q| = 2$ and hence the expression is just $-R_{i\bar{j}k\bar{l}}$. Assume that the lemma is proved for any $|P| + |Q| \leq n$. Using (10), any derivative (evaluated at z_0) of the form

$$\frac{\partial^{|C|+|D|}g_{i\bar{j}}}{\partial z^D \partial \bar{z}^C} \tag{35}$$

with $|C| + |D| = n + 1$, can be represented as

$$-\frac{\partial^{n-1}R_{i\bar{j}k\bar{l}}}{\partial z^{D'} \partial \bar{z}^{C'}} + \frac{\partial^{n-1}}{\partial z^{D'} \partial \bar{z}^{C'}}\left(g^{p\bar{q}} \cdot \frac{\partial g_{i\bar{q}}}{\partial z_k} \cdot \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}\right) \tag{36}$$

for some C', D' such that $|C'| + |D'| = n - 1$. On the other hand, any derivative of the form $\frac{\partial}{\partial z_s}R_{i\bar{j}k\bar{l},A\bar{B}}$ can be represented as a polynomial of the covariant derivatives of the curvature and the Christoffel symbols Γ_{ij}^k. Since

$$\Gamma_{ij}^k = g^{k\bar{q}} \cdot \frac{\partial g_{i\bar{q}}}{\partial z_j} \in \mathcal{R} \, ,$$

by the inductive assumption, (36) is in $\mathcal{R}$, and the first part of the lemma is proved.

To prove the second part of the lemma, we extend the weight w so that

$$w(g_{i\bar{j}}) = w(g^{i\bar{j}}) = 0, \quad w\left(\frac{\partial^{|P|+|Q|}g_{i\bar{j}}}{\partial z^P \partial \bar{z}^Q}\right) = (|P| + |Q|)/2 \, ,$$

and we extend the definition of w to monomials by requiring $w(AB) = w(A) + w(B)$. (We remark that the w-weight is well-defined and is independent to the choice of local coordinates.) Moreover, such an extension is compatible with formula (10) since each monomial in (10) is of weight 1. As a result, in the procedure of writing (33) in terms of the covariant derivatives of the curvature, at each step, we get a w-homogeneous polynomial in $\mathcal{R}$. Since

$$\frac{\partial^{P+Q}\varphi}{\partial z^P \partial \bar{z}^Q} = \frac{\partial^{|P'|+|Q'|} g_{i\bar{j}}}{\partial z^{P'} \partial \bar{z}^{Q'}}$$

for some multiple indices P', Q' such that $|P'| = |P| - 1, |Q'| = |Q| - 1$, this degree is equal to

$$w\left(\frac{\partial^{|P'|+|Q'|} g_{i\bar{j}}}{\partial z^{P'} \partial \bar{z}^{Q'}}\right) = \frac{|P'| + |Q'|}{2} = \frac{|P| + |Q|}{2} - 1.$$

At z_0, $g_{i\bar{j}} = \delta_{ij}$ and

$$\frac{\partial^{|C|} g_{i\bar{j}}}{\partial \bar{z}^C} = \frac{\partial^{|D|} g_{i\bar{j}}}{\partial z^D} = 0$$

for any $C, D \neq 0$. Thus the expression at z_0 is a w-homogeneous polynomial of the covariant derivatives of the curvature of w-weight $(|P| + |Q|)/2 - 1$.

$\square$

By the same argument, we have

Lemma 3.2. *Let F be a w-homogeneous polynomial of weight d in the covariant derivatives of the curvature. Then using K-coordinates of order n at z_0, the expression*

$$\frac{\partial^{|P|+|Q|} F}{\partial z^P \partial \bar{z}^Q}$$

is a w-homogeneous polynomial of w-weight $d + (|P| + |Q|)/2$, for $|P| + |Q| \leq n - 2d - 2$.

Proof of Theorem 1.7. Applying Lemmas 3.1 and 3.2 to formula (27), we see that β_r is a polynomial in $R_{i\bar{j}k\bar{l}, P\overline{Q}}$, $u, \bar{u}, v, \bar{v}$, for $2 \leq r \leq n - 2$. To prove the homogeneity, we further extend the w-weight by requiring $w(fN^{-r/2}) = w(f) - r/2$, where $f \in \mathcal{F}$ is a w-homogeneous polynomial and $r \in \mathbb{Z}$. A formal series is called *regular*, if all of its monomials are of weight 0. The set of regular formal series forms an algebra over $\mathbb{C}$. Then by [12, Theorem 1.1], the Catlin-Zelditch asymptotic series (8) is regular. It follows immediately that the asymptotic expansion (26) is regular, and therefore by Lemma 3.2,

$$\sum_{j=1}^{s} N^{-\frac{|P|+|Q|}{2}-j} \frac{\partial^{|P|+|Q|} \alpha_j}{\partial z^P \partial \bar{z}^Q}(z_0)$$

is regular for $s + |P| + |Q| \leq n - 2$. By Lemma 3.1,

$$\sum_{|P|+|Q|\leq n} N^{-\frac{|P|+|Q|}{2}+1} \varphi_{P\overline{Q}}(z_0)$$

is also regular. Thus by Lemma 2.6, the partial sum from (22),

$$\Sigma_r := \sum_{j=2}^{r} N^{-j/2} \beta_j$$

is regular. Therefore

$$\exp(\Sigma_r) = 1 + N^{-1}b_2 + \cdots + N^{-r/2}b_r + O(N^{-\frac{r+1}{2}}) \tag{37}$$

is a regular series, and hence $w(b_r) = r/2$.

To prove the second part of the theorem, we recall from Lemma 2.6 that β_r is a polynomial of degree at most $r + 2$ in $(u, v, \bar{u}, \bar{v})$. It then follows from (37) that the polynomial b_r is of the form

$$b_r = \sum \left\{ q_{a_1\cdots a_t} \prod_{i=1}^{t} \beta_{a_i} \ : \ 1 \le t \le \left\lfloor \frac{r}{2} \right\rfloor, \ 2 \le a_1 \le \cdots \le a_t, \ \sum_{i=1}^{t} a_i = r \right\}, \tag{38}$$

where $q_{a_1\cdots a_t} \in \mathbb{Q}$. Since $\deg \beta_{a_i} \le a_i + 2$, it follows that

$$\deg b_r \le \max_{a_1,\ldots,a_t} \sum_{i=1}^{t}(a_i + 2) = r + 2 \left\lfloor \frac{r}{2} \right\rfloor = \left\{ \begin{array}{ll} 2r, & \text{for } r \text{ even,} \\ 2r - 1, & \text{for } r \text{ odd.} \end{array} \right. \tag{39}$$

Indeed, for $r = 2s$, the sum in (38) contains the term $\frac{1}{s!}(\beta_2)^s$, which contributes $\frac{1}{s!4^s}S^\sharp(\mathbf{u}, \mathbf{v})^s$, which gives the terms of top degree $2r$ in b_r. Similarly, if $r = 2s + 1$, then b_r contains $\frac{1}{3\cdot 4^s(s-1)!}S^\sharp(\mathbf{u}, \mathbf{v})^{s-1}L^\sharp(\mathbf{u}, \mathbf{v})$, which is homogeneous of degree $4s + 1 = 2r - 1$. Thus we have equality in (39). $\qquad\square$

Remark: It follows from (27) that β_r has the same parity as r. Thus each monomial of β_{a_i} has degree equal to a_i mod 2, and (38) then gives an alternative proof that b_r also has the same parity as r.

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